Classifying plane curves of a given genus with singularities in general position

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October 3, 2005
Birational Geometry Seminar

arxiv: 2508.01462

Plane curves of genus 0 and 1

Notation

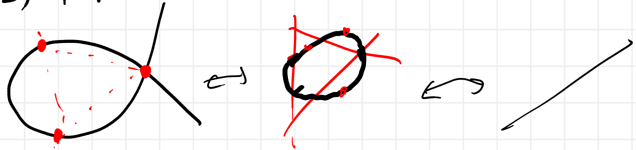
- $p_1, \dots, p_n \in \mathbb{P}^2$ general points in $\mathcal{U} \subset (\mathbb{P}^2)^n$ Zariski open
- $|d; m_1, \dots, m_n| = \{\text{Curves } C \text{ with } \deg C = d,\ \mathrm{mult}_{p_i} C \geq m_i\}$
- $g(d; m_1, \dots, m_n) = \frac{(d-1)(d-2)}{2} - \sum \frac{m_i(m_i-1)}{2}$ (genus)

Theorem (Classical)

Assume $\mathcal{L} = |d; m_1, \dots, m_n|$ contains an irreducible curve C.

- *If $g(\mathcal{L}) = 0$, C is the Cremona transform of a line*
- *If $g(\mathcal{L}) = 1$, C is the Cremona transform of a smooth cubic*

$|3;2,1|$



General points!

$\leftarrow 10$ pts

$|6;2^{10}|$ $\quad g=0 \quad$ rational nodal sextic

$\longrightarrow$ system empty $p_1 - p_{10}$ general

$|6;2^9| \quad \longrightarrow$ Halphen pencil $\quad g=1$

but if $p_1 - p_9$ general

$|\quad\quad| = 2\times$ cubic

Higher genus

Theorem (De Franchis 1899, Calabri-Ciliberto 2010)

Assume $\mathcal{L} = |d; m_1, \ldots, m_n|$ *contains an irreducible curve* C *and* $\dim \mathcal{L} \geq 1$*. If* $g = 2$*,* C *is the Cremona transform of either*

- *a nodal quartic*
- *a sextic with 8 nodes*

Main result

Theorem

Assume the SHGH conjecture holds, and fix $g \geq 2$.
Up to Cremona transform, there are finitely many numerical characters $|d; m_1, \ldots, m_n|$ for which $\mathcal{L} \neq \emptyset$ and $g(\mathcal{L}) = g$

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- $X_n \longrightarrow \mathbb{P}^2$ blowing up at $p_1, \ldots, p_n$
- $\operatorname{Pic} X_n = \langle L, E_1, \ldots, E_n \rangle_{\mathbb{Z}}$,
  $L^2 = 1, E_i^2 = -1, L \cdot E_i = E_i \cdot E_j = 0$
- $K_n = -3L + E_1 + \cdots + E_n$
- $\mathcal{L} = |d; m_1, \ldots, m_n| \cong$
  $\mathbb{P}(H^0(X_n, \mathcal{O}_{X_n}(dL - m_1 E_1 - \cdots - m_n E_n)))$
- $\mathcal{L}^2 = d^2 - \sum m_i^2$, $\mathcal{L} \cdot K_n = -3d + \sum m_i$
- (-1)-curve E: $E^2 = E \cdot K_n = -1$

Main result

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- *Expected dimension*
  $\operatorname{edim} \mathcal{L} := \max\left\{\frac{d(d+3)}{2} - \sum \frac{m_i(m_i+1)}{2}, -1\right\} \leq \dim \mathcal{L}$

Conjecture (Segre-Harbourne-Gimigliano-Hirschowitz)

If $\operatorname{edim} \mathcal{L} < \dim \mathcal{L}$ then there exists a (-1)-curve E in X_n with $E \cdot \mathcal{L} < -1$.

Main result

$$\begin{tikzcd} & X_n \arrow[dl] \arrow[dr] & \\ \mathbb{P}^2 \arrow[rr, dashed] & & \mathbb{P}^2 \end{tikzcd}$$

Theorem

Assume the SHGH conjecture holds, and fix $g \geq 2$.
Up to Cremona transform, there are finitely many numerical characters $|d; m_1, \ldots, m_n|$ for which $\mathcal{L} \neq \emptyset$ and $g(\mathcal{L}) = g$

- Cremona map dominated by $X_n \leftrightsquigarrow$ automorphism of $\operatorname{Pic} X_n$
  Standard quadratic Cremona $\leftrightsquigarrow$ reflection $[L - E_1 - E_2 - E_3]$
  $|d; m_1, m_2, m_3, m_4 \ldots| \overset{\phi}{\mapsto} |d + \delta; m_1 + \delta, m_2 + \delta, m_3 + \delta, m_4 \ldots|$
  where $\delta = d - m_1 - m_2 - m_3 = \mathcal{L} \cdot [L - E_1 - E_2 - E_3]$
- $\mathcal{CK}_n \subset SL_{n+1}(\mathbb{Z})$ group: generated by ϕ and S_n
- g, $\mathcal{L}^2$, $\dim \mathcal{L}$ are preserved by $\mathcal{CK}$

- $\mathcal{L}$ Cremona-reduced: $d \geq m_1 + m_2 + m_3$ and $m_1 \geq m_2 \geq \cdots \geq 0$

Consequences of the SHGH conjecture

- If E is an irreducible curve on X_n with $E^2 < 0$, then E is a (-1)–curve
- If F effective divisor on X_n, either nef or reduced, then
  - $h^i(X_n, F) = 0$ for $i = 1, 2$
  - $\dim \mathcal{L}_F = \operatorname{edim} \mathcal{L}_F = F^2 - g(\mathcal{L}_F) + 1$
- If F effective nef divisor on X_n, then either
  - $\mathcal{L}_F = \{C\}$ where C irreducible
  - $\mathcal{L}_F = \{mC\}$ where C irreducible, $C^2 = 0$ and $g(C) = 1$
  - $\mathcal{L}_F$ has no fixed component

Assume $\dim \mathcal{L} \geq 0$, $g(\mathcal{L}) \geq 2$ and Cremona-reduced
Then SHGH also implies:

- The general curve $C \in \mathcal{L}$ is irreducible
- The adjoint system $|C + K_n|$ has dimension $g(\mathcal{L}) - 1$ and no fixed component

Cremona-minimal classification for $g \geq 2$ assuming SHGH

Assume $|C + mK_n|$ effective, $|C + (m+1)K_n|$ not, and let $\alpha = \dim(|C + mK_n|)$

$\mathcal{L} = |C|$

(i) $|3m; m_1, \ldots, m_n|$ with $m \geq m_1 \geq \cdots \geq m_n \geq 1$; $(\alpha = 0)$

(ii) $|3m + e; m + e, m + e, m_3, \ldots, m_n|$ with
$m - e \geq m_3 \geq \cdots \geq m_n \geq 1$, $m > e > 0$; $(\alpha = 0)$

$|6; 2^8, 1|$
$m = 2$

(iii) $|3m + \alpha; m + \alpha, m_2, \ldots, m_n|$ with
$m \geq m_2 \geq \cdots \geq m_n \geq 1$; $(\alpha \geq 1)$

$\leftarrow |4; 2, 1^{n-1}|$ $\quad n \leq 11$
$m = 1, \ \alpha = 1$

(iv) $|3m + \alpha + e; m + \alpha + e, m + e, m_3, \ldots, m_n|$ with
$m - e \geq m_3 \geq \cdots \geq m_{n-1} \geq 1$, $m > e > 0$; $(\alpha \geq 1)$

(v) $|3m + \left\lfloor \frac{\alpha}{2} \right\rfloor; m_1, \ldots, m_n|$ with
$m \geq m_1 \geq \cdots \geq m_n \geq 1$; $(\alpha \in \{2, 5\})$

(vi) $|3m + \frac{\alpha}{2}; m - 1 + \frac{\alpha}{2}, m_2, \ldots, m_n|$ with
$m \geq m_2 \cdots \geq m_n \geq 1$; $(\alpha \geq 4$ even$)$

(vii) $|3m + \frac{\alpha+1}{2} + e; m + \frac{\alpha-1}{2} + e, m + e + 1, m_3, \ldots, m_n|$ with
$m - e \geq m_3 \geq \cdots \geq m_n \geq 1$ and $m > e \geq 0$; $(\alpha \geq 3$ odd$)$

Proof of the classification assuming SHGH

Lemma

The Zariski decomposition of $|C + tK_n|$ effective (not nef) is

$$C + tK_n \sim \underbrace{P}_{\text{nef}} + A, \qquad \text{where}$$

$P \cdot A_i = 0$

- $A = \sum_{i=1}^{h} (t - e_i) A_i$, $h \leq n$
- A_i *are disjoint* (-1)*-curves;* $e_i = C \cdot A_i$
- *Let* $f : X_n \to S$ *be the contraction of* A
  - $P = f^*(C_S + \cancel{n}\,t K_S)$
  - $S \cong X_{n-h}$ *or* [$S \cong \mathbb{P}^1 \times \mathbb{P}^1$ *and* $h = n - 1$]

Proof of main theorem assuming SHGH

- It is enough to show that $\{(m(\mathcal{L}), \alpha(\mathcal{L}))\}$ finite for fixed g
- It is enough to consider bounded selfintersection (Explicitly: by Ciliberto–Dedieu–Mendes-Lopes, arxiv 2503.22299)
  - if $m(\mathcal{L}) \leq 2$ then finitely many possibilities
  - if $\mathcal{L}^2 > 3g-3$ then $m(\mathcal{L}) \leq 2$
- If $n \leq 9$, $-K_n$ effective, $C \cdot K_n < 0$ gives $m(\mathcal{L}) \leq 2g-1$, $\alpha(\mathcal{L}) \leq 2g-2$ if $\mathcal{L}^2 \leq 3g-3$
- If $\mathcal{L}$ not Cremona transform of a system with ≤ 9 points use SHGH and $\mathcal{L}^2 \leq 3g-3$ to estimate $\dim |C + tK_n|$ and show

$$m(\mathcal{L}) \leq g + 11 + \sqrt{(g+11)^2 - 4(g-1)}$$

$$\alpha(\mathcal{L}) \leq (m(\mathcal{L}) + 2)(g-1) + 9m$$

$$(m(\mathcal{L}) - 2)(g-1) + 9m(\mathcal{L})$$

Bounds on selfintersection assuming SHGH

Theorem

Assume $n \geq 10$, $g(\mathcal{L}) \geq 2$, $r = \dim(\mathcal{L}) \geq 0$.

(a) *If $|C + K_n|$ composed with a pencil, then $|C|$ is Cremona equivalent to either*

(i) $|6; 2^8, 1^{n-8}|$, $8 \leq n \leq 11$ *in which case $g = 2$ and $C^2 = 12 - n$;*

(ii) $|g + 2; g, 1^{n-1}|$, $1 \leq n \leq 3g + 6$ *in which case $C^2 = 4g - n + 5 = (n + 4r)/3 - 2$*

(iii) $|9; 3^8, 2^2|$, *for which $n = 10$, $r = 0$, $g = 2$, and $C^2 = 1$.*

(b) *If $g \geq 3$, $n + r = 2h$, and $|C + K_n|$ has irreducible general member, then $C^2 \geq h + r - 5$.* $\sim \frac{n+3r}{2}$ $h = \frac{n+r}{2}$ $g' \geq 0$

(c) *If $g \geq 3$, $n + r = 2h + 1$, and $|C + K_n|$ has irreducible general member, then $C^2 \geq \lceil (6h - 7)/5 \rceil + r - 5$* $\sim \frac{3n+8r}{5}$ $g' \geq 1$

Remark

If $C' \in |C + K_n|$ irreducible of genus g', then $C^2 = \frac{n+3r+g'}{2} - 5$.

Applications assuming SHGH, for given $n \geq 10$

- Explicit list of effective Cremona-minimal systems with $\mathcal{L}^2 \leq 5$
- Classification of minimal degree maps to the plane ($\dim \mathcal{L} = 2$, $\mathcal{L}^2$ minimal) ← fixed n
- For every $\dim \mathcal{L} \geq 2$ the systems with minimal $\mathcal{L}^2$ are base point free
- For every $\dim \mathcal{L} \geq 3$ the systems with minimal $\mathcal{L}^2$ determine birational morphisms
- We expect that for every $\dim \mathcal{L} \geq 5$ the systems with minimal $\mathcal{L}^2$ linear systems on X_n are very ample. True for $n = 10$.

Thank you!